Taming the $\epsilon$—expansion with the Analytic Bootstrap

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Based on work with Henriksson and van Loon
What will this talk be about?

We will study conformal field theories in $D > 2$

- Very relevant for Physics.
- Interesting interplay with Mathematics.
- Ubiquitous in dualities in string and gauge theory.

Unfortunately, studying CFT in $D > 2$ is not so easy...

- Symmetries are less powerful than in $D = 2$...
- In general they do not have a Lagrangian description...
- In a Lagrangian theory we can use Feynman diagrams:

$$A(g) = A^{(0)} + gA^{(1)} + \cdots$$

- But generic CFTs don’t have a small coupling constant!

In spite of all this, progress can be made!

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Taming the $\epsilon$—expansion with the Analytic Bootstrap
Conformal bootstrap: resort to consistency conditions!
- Conformal symmetry
- Properties of the OPE
- Unitarity
- Crossing symmetry

Successfully applied to 2d in the eighties... 25 years later it was finally implemented in $D > 2!$ [Rattazzi, Rychkov, Tonni, Vichi '08]

The starting point of a numerical revolution.

Today: Analytic results for CFTs in the spirit of the conformal bootstrap!
Analytic conformal bootstrap

Which kind of analytic results will you get today?

**Analytic bootstrap** (or the light-cone bootstrap on esteroids)

- Results for operators with spin in a generic CFT!

\[ \mathcal{O}_\ell \sim \varphi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi \]

- Study their scaling dimension \( \Delta \) for large values of the spin \( \ell \):

\[ \Delta(\ell) = \ell + c_0 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots, \quad \text{double-twist} \]

- Obtain analytic results to all orders in \( 1/\ell \) resorting only to consistency conditions.

- These results will actually be valid even for finite values of the spin!

- Other analytic approaches to be discussed across the week!
CFT - Ingredients

Main ingredient:

- Conformal Primary local operators: \( \mathcal{O}_{\Delta,\ell}(x) \),
  \[ [K_\mu, \mathcal{O}(0)] = 0 \]
  Dimension Lorentz spin

In addition we have descendants \( P_{\mu_k} \ldots P_{\mu_1} \mathcal{O}_{\Delta,\ell} = \partial_{\mu_k} \ldots \partial_{\mu_1} \mathcal{O}_{\Delta,\ell} \).

Operators form an algebra (OPE)

\[
\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k \in \text{prim.}} C_{ijk} |x|^{|\Delta_k| - |\Delta_i| - |\Delta_j|} \left( \mathcal{O}_k(0) + x^\mu \partial_\mu \mathcal{O}_k(0) + \cdots \right)
\]

- **CFT data**: The set \( \Delta_i \) and \( C_{ijk} \) characterizes the CFT.
The observable: Four-point function of identical operators.

\[ \langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = \frac{G(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}, \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \]

It admits a decomposition in conformal blocks:

\[ \langle OOOO \rangle = \sum_{\Delta, \ell} C_{\Delta, \ell} C_{\Delta, \ell} \]

\[ G(u, v) = 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(u, v) \]

Identity operator \quad Conformal primaries \quad Conformal blocks
Conformal bootstrap

It is crossing symmetric: \( v^\Delta \phi G(u, v) = u^\Delta \phi G(v, u) \)

Combining the two...

\[
\sum_{\Delta, \ell} C_{\Delta, \ell} \mathcal{O}_{\Delta, \ell} = \sum_{\Delta, \ell} C_{\Delta, \ell} \mathcal{O}_{\Delta, \ell}
\]

A remarkable...but hard equation!

\[
v^\Delta \phi \left( 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(u, v) \right) = u^\Delta \phi \left( 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(v, u) \right)
\]

Easy to expand around \( u = 0, v = 1 \) 

Easy to expand around \( u = 1, v = 0 \)
Study this equation in different regions, $u = z\bar{z}$, $v = (1 - z)(1 - \bar{z})$

- In the Euclidean regime $\bar{z} = z^*$. 
- We can study crossing around $u = v = \frac{1}{4}$
- Starting point of the numerical bootstrap.

- In the Lorentzian regime $z, \bar{z}$ are independent real variables and we can consider $u, v \to 0$.
- Starting point of the analytic (light-cone) bootstrap!
Analytic bootstrap

Why is this a good idea?

\[ v^{\Delta} \left( 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(u, v) \right) = u^{\Delta} \left( 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(v, u) \right) \]

Direct channel \iff Crossed channel

Very complicated interplay between l.h.s. and r.h.s. \ldots but:

Operators with large spin \iff Identity operator
Conformal blocks - technicalities

- Eigenfunctions of a Casimir operator
  
  \[ C G_{\Delta,\ell}(u, v) = J^2 G_{\Delta,\ell}(u, v) \]

  where \( J^2 = (\ell + \Delta)(\ell + \Delta - 1) \sim \ell^2 \)

- Small \( u \) limit:
  
  \[ G_{\Delta,\ell}(u, v) \sim u^{\tau/2} f_{\tau,\ell}(v), \quad \tau = \Delta - \ell \]

  We will introduce the notation
  
  \[ G_{\Delta,\ell}(u, v) \equiv u^{\tau/2} f_{\tau,\ell}(u, v) \]

- Small \( v \) limit:
  
  \[ f_{\tau,\ell}(u, v) \sim \log v \]
Necessity of a large spin sector

- Consider the $v \ll 1$ limit of the crossing equation: $C_{\Delta,\ell}^2 \rightarrow a_{\tau,\ell}$

$$v^{\Delta_O} \left(1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u, v)\right) = u^{\Delta_O} \left(1 + \sum_{\tau,\ell} a_{\tau,\ell} v^{\tau/2} f_{\tau,\ell}(v, u)\right)$$

$$\Downarrow$$

$$1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u, v) = \frac{u^{\Delta_O}}{v^{\Delta_O}} + \text{subleading terms}$$

- The r.h.s. is divergent as $v \rightarrow 0$.
- Each term on the l.h.s. diverges as $f_{\tau,\ell}(u, v) \sim \log v$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_O$ (actually $\tau_n = 2\Delta_O + 2n$)
Consider the $\nu \ll 1$ limit of the crossing equation:

$$
\nu^{\Delta_O} \left( 1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u, \nu) \right) = u^{\Delta_O} \left( 1 + \sum_{\tau,\ell} a_{\tau,\ell} v^{\tau/2} f_{\tau,\ell}(v, u) \right)
$$

$$
\downarrow
$$

$$
1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u, \nu) = \frac{u^{\Delta_O}}{\nu^{\Delta_O}} + \text{subleading terms}
$$

rest of operators sorted by twist

- The r.h.s. is divergent as $\nu \to 0$.
- Each term on the l.h.s. diverges as $f_{\tau,\ell}(u, \nu) \sim \log \nu$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_O$ (actually $\tau_n = 2\Delta_O + 2n$)
Example: Generalised free fields

- Simplest solution: Generalised free fields

\[ G^{(0)}(u, v) = 1 + \left( \frac{u}{v} \right)^{\Delta_\phi} + u^{\Delta_\phi} \]

- Intermediate ops: Double twist operators:

\[ \mathcal{O} \Box^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O} \]

\[ \tau_{n,\ell} = 2\Delta_\mathcal{O} + 2n \]

\[ a_{n,\ell} = a_{n,\ell}^{(0)} \]

- Their OPE coefficients are such that the divergence of a single conformal block (\( \sim \log v \)), as \( v \to 0 \), is enhanced!

\[ 1 + \sum_{\tau,\ell} a_{\tau_n,\ell}^{(0)} u^{\tau_n/2} f_{\tau_n,\ell}(u, v) = 1 + \left( \frac{u}{v} \right)^{\Delta_\phi} + u^{\Delta_\phi} \]

But this divergence is quite universal!
Additivity property [Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov]

In any CFT with $\mathcal{O}$ in the spectrum, crossing symmetry implies the existence of double twist operators with arbitrarily large spin and

\[
\begin{align*}
\tau_{n,\ell} &= 2\Delta_\mathcal{O} + 2n + \mathcal{O}\left(\frac{1}{\ell}\right) \\
a_{n,\ell} &= a_{n,\ell}^{(0)} \left(1 + \mathcal{O}\left(\frac{1}{\ell}\right)\right)
\end{align*}
\]

- All CFTs have a large spin sector, for which the operators become “free”!
- Can we do perturbations around large spin? YES! We will develop a large spin perturbation theory.
Large spin perturbation theory

- We would like to exploit the following idea

\[ \sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) = \frac{u^{\Delta_\mathcal{O}}}{v^{\Delta_\mathcal{O}}} + \cdots \]

Behaviour at large spin \( \Leftrightarrow \) Enhanced divergences as \( v \to 0 \)

- The presence of the identity on the r.h.s. already led to a remarkable result.
- Let's take this to the next level! Solve the following problem:

Find \( a_{\tau, \ell} \) such that

\[ \sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) = \text{Given enhanced singularity} \]

Power law divergent after applying \( \mathcal{C} \) finite times
Large spin perturbation theory

- GFF has accumulation points in the twist at
  \[ \tau_n = 2\Delta_\mathcal{O} + 2n \]

- It is convenient to define "twist" conformal blocks, the contribution to the 4pt function from a given twist.
  \[ \sum_{\ell} a_{\tau,\ell}^{(0)} u^{\tau/2} f_{\tau,\ell}(u, v) \equiv H_{\tau}(u, v) \]

- And twist conformal blocks with insertions:
  \[ \sum_{\ell} a_{\tau,\ell}^{(0)} \frac{u^{\tau/2}}{J^{2m}} f_{\tau,\ell}(u, v) \equiv H_{\tau}^{(m)}(u, v) \]
Large spin perturbation theory

Properties

▷ Correlator decomposition

\[ G^{(0)}(u, v) = \sum_n H^{(0)}_{\tau n}(u, v) \]

▷ Recurrence relation

\[ C H^{(m+1)}_{\tau}(u, v) = H^{(m)}_{\tau}(u, v) \]

▷ Prescribed behaviour at small \( u \) and small \( v \).

\[ H^{(m)}_{\tau}(u, v) \sim u^{\tau/2}, \quad \text{small } u \]
\[ H^{(m)}_{\tau}(u, v) \sim v^{-\Delta_\phi + m}, \quad \text{small } v \]

The functions \( H^{(m)}_{\tau}(u, v) \) can be systematically constructed!
Back to our problem!

\[ \sum_{\tau, \ell} a^{\tau/2}_{\tau, \ell} f_{\tau, \ell}(u, v) = \text{Sing}(u, v) \]

Assume \( a_{\tau, \ell} \) admits the following expansion

\[ a_{\tau, \ell} = a^{(0)}_{\tau, \ell} \left( b_{\tau, 0} + \frac{b_{\tau, 1}}{J^2} + \frac{b_{\tau, 2}}{J^4} + \cdots \right) \]

\[ \Downarrow \]

\[ \sum_{\tau, m} b_{\tau, m} H^{(m)}_\tau(u, v) = \text{Sing}(u, v) \]

Expanding both sides for small \( u, v \) we can find all coefficients \( b_{\tau, m} \)!

The problem has become algebraic!
The 'enhanced-singularity' is equivalent to the double-discontinuity around $v = 0$:

$$d\text{Disc}[\log^2 v] \sim 1, \quad d\text{Disc}\left[\frac{1}{v^p}\right] \sim \frac{1}{v^p} \sin^2 p\pi, \quad d\text{Disc}[f_{\tau,\ell}(u, v)] = 0$$

Consider the inversion problem for a given twist

$$\sum_{\ell} a_{\ell} f_{\tau,\ell}^{\text{coll}}(v) = G(v)$$

$$\Downarrow$$

$$a_{\ell} = \int_{0}^{1} dv \ K(\ell, v) \ d\text{Disc}[G(v)]$$

LSPT implies an equation for $K(\ell, v)$ and fixes it.

The two methods are equivalent, but the latter is explicitly analytic in the spin!
Wider perspective on CFT

- Large spin perturbation theory allows to reconstruct the CFT-data from the enhanced singularities, but... the structure of singularities can be extremely complicated!

If two operators $\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}$ of twists $\tau_1$ and $\tau_2$ are part of the spectrum then there is a tower of operators $[\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}]_{n,\ell}$ of twist

$$
\tau[\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}]_{n,\ell} = \tau_1 + \tau_2 + 2n + \mathcal{O}\left(\frac{1}{\ell}\right)
$$

- This should make you happy and sad at the same time!

The spectrum of generic CFTs is hard!

- $\mathcal{O}$ is part of the spectrum.
- $[\mathcal{O}, \mathcal{O}]_{n,\ell}$ is also part of the spectrum.
- And $[[\mathcal{O}, \mathcal{O}]_{n_1,\ell_1}, [\mathcal{O}, \mathcal{O}]_{n_2,\ell_2}]_{n_3,\ell_3}$ too, and so on!
In non-perturbative CFTs the spectrum is very rich. Hard (but not impossible!) to apply our idea

\[ \sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) = \text{Rich spectrum in the crossed channel} \]

Behaviour at large spin \( \Leftrightarrow \) complicated divergences as \( v \to 0 \)

- If the CFT has a small parameter we are better of, as this parameter further organises the problem.
**LSPT: Strategy**

**Strategy**

1. Use crossing symmetry to determine the enhanced singularities

\[
G(u, v) \leftarrow G(u, v)|_{en.sing.} = \left( \frac{u}{v} \right)^{\Delta_0} G(v, u)|_{en.sing.}
\]

In theories with small parameters the latter follows from CFT-data at lower orders! (maybe including other correlators)

2. Then use LSPT to reconstruct the CFT-data from the enhanced singularities.

3. Go to next order and repeat.

This can be turn into an efficient machinery!

- Theories with weakly broken higher spin symmetry.
- Weakly coupled gauge theories.
- $1/N^4$ corrections to GFF and $\mathcal{N} = 4$ SYM.
CFT with weakly broken HS symmetry

CFT with HS symmetry
- The following is part of the spectrum
  - Fundamental scalar field $\phi$
    \[
    \partial_\mu \partial^\mu \phi = 0 \rightarrow \Delta_\phi = \frac{d - 2}{2}
    \]
  - Tower of HS conserved currents $J^{(s)} = \phi \partial_{\mu_1} \cdots \partial_{\mu_s} \phi$,
    \[
    \text{conservation} \rightarrow \Delta_s = d - 2 + s
    \]

Weakly broken HS symmetry
- The fundamental field $\phi$ and HS currents $J^{(s)}$ acquire an anomalous dimensions:
  \[
  \Delta_\phi = \frac{d - 2}{2} + g \gamma_\phi + \cdots
  \]
  \[
  \Delta_s = d - 2 + s + g \gamma_s + \cdots
  \]

Which anomalous dimensions are consistent with crossing symmetry?
We will study the WF-model in $d = 4 - \epsilon$ dimensions, where $g \sim \epsilon$.

$$\Delta_\varphi = \frac{d - 2}{2} + \epsilon \gamma^{(1)}_\varphi + \cdots$$

$$\Delta_s = d - 2 + s + \epsilon \gamma^{(1)}_s + \cdots$$

Carry our program for the correlator $\langle \varphi \varphi \varphi \varphi \rangle$

$$\mathcal{G}(u, v) = 1 + \left( \frac{u}{v} \right)^{\Delta_\varphi} + u^{\Delta_\varphi} + \epsilon \mathcal{G}^{(1)}(u, v) + \cdots$$

To leading order the intermediate operators are the identity and the tower of conserved currents $J^{(s)}$. 
Look at the contribution from the currents to the correlator

\[ G(u, v) \big|_{\text{currents}} = \sum_{\ell} a_{\ell} u^{\frac{d-2}{2}} + \epsilon \frac{\gamma_{\ell}}{2} f_{\frac{d-2}{2}} + \epsilon \frac{\gamma_{\ell}}{2}, \ell(u, v) \]

\[ = \sum_{\ell} a^{(0)}_{\ell} u^{\frac{d-2}{2}} f_{\frac{d-2}{2}}, \ell(u, v) + \epsilon \sum_{\ell} u^{\frac{d-2}{2}} \left( a^{(0)}_{\ell} \frac{\gamma_{\ell}}{2} (\log u + \partial_{\tau}) + a^{(1)}_{\ell} \right) f_{\frac{d-2}{2}}, \ell(u, v) \]

\[ + \cdots \]

Use crossing symmetry to determine the enhanced singularities in an \( \epsilon \)–expansion.

Use the previous method to compute

\[ a^{(0)}_{\ell}, \quad a^{(0)}_{\ell} \gamma_{\ell}, \quad a^{(1)}_{\ell}, \cdots \]
Determining the enhanced singularities in a $\epsilon$–expansion...

- The operator $\phi^2$ is special because it's the only one that gets a correction to order $\epsilon$.

$$G(u, v)|_{\text{enh.sing.}} = \left( \frac{u}{v} \right)^{\Delta \phi} \left( \underbrace{1}_{\text{identity}} + a_{\phi^2} v^{\Delta \phi^2/2} f_{\phi^2}(v, u) \right) + \cdots$$

- To order $\epsilon^3$ nothing else contributes a enhanced singularity!

- The method we described allows to compute the anomalous dimensions and OPE coefficients of $J^{(s)}$ to this order!
An ugly fact

- To order $\epsilon^2$ new, quartic, intermediate operators appear

$$\varphi^2 \Box^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi^2$$

- This has the potential to make our lives miserable at order $\epsilon^4$...

A pleasant surprise

- With our procedure we can compute the OPE coefficient with which they appear.
- Only operators with $n = 0$ (approximate twist four) appear! [Conjectured some time ago by Liendo, Rastelli and van Rees]
- This will simplify our life!
To order $\epsilon^4$ two new sources of enhanced-singularities appear

1. Weakly broken currents in the dual channel $(I_2)$

$$a_s \sim \epsilon^0, \quad \gamma_s \sim \epsilon^2 \rightarrow a_s v^{\gamma_s/2} \sim a_s \gamma_s^2 \log^2 v \sim \epsilon^4$$

2. New quadrilinear operators in the dual channel $(I_4)$

$$a_s \sim \epsilon^2, \quad \gamma_s \sim \epsilon \rightarrow a_s^{(q)} v^{\gamma_s^{(q)}/2} \sim a_s \gamma_s^2 \log^2 v \sim \epsilon^4$$

The computation of $I_2$ is straightforward. The computation of $I_4$ is vastly simplified by the observation we made in the previous slide!
Structure of enhanced-singularities to order $\epsilon^4$

\[
G(u, v)\big|_{\text{enh. sing.}} = \left(\frac{u}{v}\right)^{\Delta_{\varphi}} \left(1 + a_{\varphi^2} v^{\Delta_{\varphi^2}/2} f_{\varphi^2}(v, u) + l_2(v, u) + l_4(v, u)\right) + \cdots
\]

- We can compute the CFT-data to order $\epsilon^4$!

\[
\frac{C_T}{C_{\text{free}}} = 1 - \frac{5}{324} \epsilon^2 - \frac{233}{8748} \epsilon^3 - \left(\frac{100651}{3779136} - \frac{55}{2916} \zeta_3\right) \epsilon^4 + \cdots
\]
A transcendentality principle for WF?

\[
G_{\text{conn}}(u, v)|_{\text{e.s.}} \sim \epsilon^2 \log^2 v (2 \log(1 - v) - \log u) + \\
\epsilon^3 \log^2 v (\log u + \log u \log(1 - v) + Li_2 v + \cdots) + \\
\epsilon^4 \log^2 v (\log^2 u \log(1 - v) + Li_3 v + \cdots) + \cdots
\]

- The double discontinuity contributing to the currents is formed by pure transcendental functions!
- With this assumption one can go even higher in the $\epsilon$ expansion!
Conclusions

- Generic CFTs have a large spin sector, which becomes essentially free. We have shown how to perform a perturbation around that sector.
- This applies to vast families of CFTs and provides an alternative to diagrammatic computations.
- Do we have a transcendentality principle for WF?
- There are many other analytic approaches to CFT, in the spirit of the conformal bootstrap, and the connection to these is a fascinating question.