Neutrino Masses and Flavor Oscillations

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★ Neutrino mass generation
★ Lepton flavor mixing issue
★ Neutrino flavor oscillation
★ Matter effects in $\nu$-oscillations
★ $\nu$-masses in $\beta$ and $0\nu2\beta$ decays
★ On the sterile neutrino species

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Lecture B

★ Matter effects in $\nu$-oscillations
★ $\nu$-masses in $\beta$ and $0\nu2\beta$ decays
★ On the sterile neutrino species
MSW matter effects

☆ When a neutrino beam propagates through a medium, the three $\nu$ flavors can interact with the quarks in the nucleons and the electrons in the atoms via elastic and inelastic scatterings. Inelastic scattering and elastic scattering off the forward direction will cause attenuation of the neutrino beam. But the cross sections are tiny, so attenuation effects are insignificant. Elastic/coherent forward scattering matters, because it may modify the vacuum behaviors of neutrino oscillations. This is just the well-known matter effect (L. Wolfenstein 1978).

Matter effects inside the Sun may enhance solar neutrino oscillations (S.P. Mikheyev and A.Yu. Smirnov 1985—MSW effect); matter effects inside the Earth can cause a day-night effect. Matter effects in long baseline experiments may result in fake CP violation (A.K. Ichikawa).
Hamiltonian in vacuum

In vacuum the evolution of three neutrino mass eigenstates with time

$$i \frac{d}{dt} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = H_0 \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad H_0 = \frac{1}{2E} \begin{pmatrix} m_1^2 & m_1 m_2 & m_1 m_3 \\ m_2 m_1 & m_2^2 & m_2 m_3 \\ m_3 m_1 & m_3 m_2 & m_3^2 \end{pmatrix}, \quad \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

In the flavor basis the evolution of three neutrino flavors is described by the Schroedinger-like equation:

$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U H_0 U^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Propagating in a medium, neutrinos may have CC and NC interactions:

- CC
  - $e^- \rightarrow \nu_e \rightarrow \nu_e$
  - $e^- \rightarrow \nu_e \rightarrow e^-$

- NC
  - $\nu_e \rightarrow \nu_e$
  - $\nu_e \rightarrow \nu_e$ (electron replaced by proton or neutron)
Matter potential

In this case the effective Hamiltonian with a matter potential given by

\[ H_m = U H_0 U^\dagger + \begin{pmatrix} V_{CC} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} V_{NC} & V_{NC} \\ V_{NC} & V_{NC} \end{pmatrix} \]

from electron \hspace{1cm} \text{from neutron}

The NC contributions from electrons and protons cancel each other, since we stay with normal matter:

\[
\begin{align*}
N_e &= N_p \\
V_{CC} &= +\sqrt{2} G_F N_e \quad \text{OK} \\
V_{NC}^n &= -\frac{1}{\sqrt{2}} G_F N_n \quad \text{OK} \\
V_{NC}^p &= +\frac{1}{\sqrt{2}} G_F N_p \left(1 - 4 \sin^2 \theta_w \right) \\
V_{NC}^e &= -\frac{1}{\sqrt{2}} G_F N_e \left(1 - 4 \sin^2 \theta_w \right)
\end{align*}
\]

\* The NC term is universal for three neutrino flavors, and hence it can be neglected in the standard case.

\* When an antineutrino beam is taken into consideration, the CC and NC terms flip their signs, and simultaneously the flavor mixing matrix \( U \) needs to be complex conjugated.

\* The NC term should not be ignored if sterile neutrinos are included.
In a medium, the correction to behaviors of neutrino oscillations is matter effects from the CC-induced coherent forward scattering.

\[
H_m = \frac{1}{2E} \left[ U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \equiv \frac{1}{2E} V \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} V^\dagger
\]

Using the effective quantities defined in matter, one may write out neutrino oscillation probabilities in the same form as in vacuum!
The above form invariance reminds us of the renormalization-group equations, which serve as a powerful tool for the systematical study of the changes of a physical system as viewed at different distance / energy scales.

As the scale varies, it is as if one is changing the magnifying power of a microscope viewing the system, which at one scale will generally be seen to consist of self-similar copies of itself viewed at another scale.


2 Nobel Prizes in condensed-matter physics (Kenneth Wilson 1982) and particle physics (David Gross, Frank Wilczek, David Politzer 2004)
Renormalization-group running

To every man is given the key to the gates of heaven. The same key opens the gates of hell.

R. Feynman

★ The point: quantities in vacuum can be extrapolated from in matter

S.H. Chiu, T.K. Kuo, 1712.08487
Z.Z. Xing, S. Zhou, 1802.00990

- Planck scale: $\Lambda \sim 10^{19}\,\text{GeV}$
- GUT scale?: $\Lambda \sim 10^{16}\,\text{GeV}$
- Seesaw scale?: $\Lambda \sim 10^{12}\,\text{GeV}$
- TeV / SUSY?: $\Lambda \sim 10^3\,\text{GeV}$
- Fermi scale: $\Lambda \sim 10^2\,\text{GeV}$
- QCD scale: $\Lambda \sim 10^2\,\text{MeV}$

- RGE-like running
- Vacuum $\alpha$
- Matter

★ $\frac{d}{d\alpha} \tilde{J}$
- $\frac{d}{d\alpha} |V_{\alpha i}|^2$
- $\frac{d\tilde{m}_i^2}{d\alpha}$
The 2-flavor case (1)

The effective Hamiltonian for two-flavor neutrinos in vacuum/matter:

\[ \mathcal{H}_v = \frac{1}{2E} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

in vacuum

\[ \mathcal{H}_m = \frac{1}{2E} \begin{pmatrix} \cos \tilde{\theta} & \sin \tilde{\theta} \\ -\sin \tilde{\theta} & \cos \tilde{\theta} \end{pmatrix} \begin{pmatrix} \tilde{m}_1^2 & 0 \\ 0 & \tilde{m}_2^2 \end{pmatrix} \begin{pmatrix} \cos \tilde{\theta} & -\sin \tilde{\theta} \\ \sin \tilde{\theta} & \cos \tilde{\theta} \end{pmatrix} \]

in matter

\[ \begin{pmatrix} m_1^2 + m_2^2 - \Delta m^2 \cos 2\theta + 4\sqrt{2}G_F N_e E & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & m_1^2 + m_2^2 + \Delta m^2 \cos 2\theta \end{pmatrix} \]

The 2-flavor approximation is good for solar or atmospheric neutrinos.
Effective neutrino mass-squared difference & mixing angle in matter:

\[ \Delta \tilde{m}^2 \sin 2\tilde{\theta} = \Delta m^2 \sin 2\theta \]

\[ \tilde{m}_1^2 + \tilde{m}_2^2 + \Delta \tilde{m}^2 \cos 2\tilde{\theta} = m_1^2 + m_2^2 + \Delta m^2 \cos 2\theta \]

\[ \tilde{m}_1^2 + \tilde{m}_2^2 - \Delta \tilde{m}^2 \cos 2\tilde{\theta} = m_1^2 + m_2^2 - \Delta m^2 \cos 2\theta + 4\sqrt{2}G_F N_e E \]

\[ P(\nu_e \rightarrow \nu_\mu)_v = \sin^2 2\theta \sin^2 \left( \frac{1.27 \Delta m^2 L}{E} \right) \]

\[ P(\nu_e \rightarrow \nu_\mu)_m = \sin^2 2\tilde{\theta} \sin^2 \left( \frac{1.27 \Delta \tilde{m}^2 L}{E} \right) \]

The matter density changes for solar neutrinos to travel from the core to the surface:

\[
\begin{pmatrix}
|\nu_e\rangle \\
|\nu_\mu\rangle
\end{pmatrix} = \begin{pmatrix}
\cos \tilde{\theta} & \sin \tilde{\theta} \\
-\sin \tilde{\theta} & \cos \tilde{\theta}
\end{pmatrix} \begin{pmatrix}
|\tilde{\nu}_1\rangle \\
|\tilde{\nu}_2\rangle
\end{pmatrix}
\]

\[ \Delta \tilde{m}^2 = \sqrt{\left( \Delta m^2 \cos 2\theta - 2\sqrt{2} G_F N_e E \right)^2 + \left( \Delta m^2 \sin 2\theta \right)^2} \]

\[ \tan 2\tilde{\theta} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2} G_F N_e E} \]

exercise: discuss 2 extreme cases:

\[ \tilde{\theta} = \frac{\pi}{2} \text{ or } \frac{\pi}{4} \]
Example: solar neutrinos

Since R. Davis’ pioneering experiment in 1968, the solar neutrino flux has been observed at Super-K (1998), SNO (2001), Borexino (2007)...

Examples: Boron (硼) ν’s ~ 32%, Beryllium (铍) ν’s ~ 56%
In the two-flavor approximation, the effective Hamiltonian of solar neutrinos is:

\[ \mathcal{H}_{\text{eff}} = \frac{\Delta m^2_{21}}{4E} \begin{bmatrix} -\cos 2\theta_{12} & \sin 2\theta_{12} \\ \sin 2\theta_{12} & \cos 2\theta_{12} \end{bmatrix} + \begin{bmatrix} \sqrt{2}G_FN_e(r) & 0 \\ 0 & 0 \end{bmatrix} \]

- Vacuum contribution: \(7.6 \times 10^{-5} \text{ eV}^2\)
- Matter correction: \(0.75 \times 10^{-5} \text{ eV}^2 / \text{MeV} (\text{at } r = 0)\)

**Be-7 \(\nu\)'s:** \(E \sim 0.862 \text{ MeV}\). The vacuum term is dominant. The survival probability on the earth is (for \(\theta_{12} \sim 34^\circ\)):

\[ P(\nu_e \rightarrow \nu_e) \approx 1 - \frac{1}{2} \sin^2 2\theta_{12} \approx 0.56 \]

**B-8 \(\nu\)'s:** \(E \sim 6 \text{ to } 7 \text{ MeV}\). The matter term is dominant. The produced \(\nu\) is roughly \(\nu_e \sim \nu_2\) (for \(V > 0\)). The \(\nu\)-propagation from the center to the outer edge of the Sun is approximately adiabatic. That is why it keeps to be \(\nu_2\) on the way to the surface (for \(\theta_{12} \sim 34^\circ\)):

\[ |\nu_2\rangle \approx \sin \theta_{12} |\nu_e\rangle + \cos \theta_{12} |\nu_\mu\rangle \]

\[ P(\nu_e \rightarrow \nu_e) = |\langle \nu_e | \nu_2 \rangle|^2 = \sin^2 \theta_{12} \approx 0.32 \]
Solar neutrino oscillations

Vacuum dominant

\[ P(\nu_e \rightarrow \nu_e) \simeq 1 - \frac{1}{2} \sin^2 2\theta_{12} \]

Matter dominant

\[ P(\nu_e \rightarrow \nu_e) \simeq |\langle \nu_e | \nu_2 \rangle|^2 \simeq \sin^2 \theta_{12} \]

\[ \theta_{12} \simeq 34^\circ \]

NOTE: terrestrial matter matters even in the JUNO reactor experiment
Open questions

We’ve learnt a lot from $\nu$ oscillations:

$$\Delta m_{21}^2, |\Delta m_{31}^2|, \theta_{12}, \theta_{13}, \theta_{23}$$

It’s more exciting that the SM is incomplete, although the Higgs has been discovered.

But a number of burning questions:

- the Majorana nature?
- the absolute $\nu$ mass scale?
- the $\nu$ mass hierarchy?
- the octant of $\theta_{23}$?
- the Dirac phase $\delta$?
- the Majorana phases?
- the sterile neutrino species?
How to weigh neutrinos?

The absolute $\nu$-mass scale?

- via the $\beta$ decays;
- via $0\nu2\beta$ decays;
- via CMB and LSS;
- via atoms and...?

K. Tsumura, K. Imamura

K. Nagamine

The most stringent constraint from cosmology (PLANK2018)

$$m_1 + m_2 + m_3 < 0.12 \text{ eV (at 95\% C.L.)}$$

Two options of the neutrino mass spectrum indicated by a global fit of current neutrino oscillation data (I. Esteban et al, 1811.05487)

<table>
<thead>
<tr>
<th>Normal ($m_1 &lt; m_2 &lt; m_3$)</th>
<th>Inverted ($m_3 &lt; m_1 &lt; m_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta m^2_{21} \times 10^{-5} \text{ eV}^2$</td>
<td>$+7.39 \text{ (bf)}$</td>
</tr>
<tr>
<td>$\Delta m^2_{31} \times 10^{-3} \text{ eV}^2$</td>
<td>$+2.52 \text{ (bf)}$</td>
</tr>
</tbody>
</table>

Normal mass ordering is favored over the inverted one at the $3\sigma$ level.
The $\beta$-decays

Three-Body Final State

1930

Helium-3 (1, 2)

1933

I will be remembered for this paper

Tritium (2, 1)

Neutron Beta Decay

Electron and neutrino share the available energy.

Initial state

Final state

$\beta$-

$N, Z \rightarrow (N - 1, Z + 1) + e^- + \bar{\nu}$.

$4p \rightarrow ^4\text{He} + 2e^+ + 2\nu_e + 26.73$ MeV

Why the Sun shines? Because

$2e^+ + 2\nu_e$

1939—1941
Today’s sensitivity

The energy spectrum of the emitted electron

\[
\frac{dN}{dE} \propto (Q - E) \sum_{i=1}^{3} |U_{ei}|^2 \sqrt{(Q - E)^2 - m_i^2}
\]

\[
\propto (Q - E)^2 \left[ 1 - \frac{\langle m_e^2 \rangle}{2 (Q - E)^2} \right]
\]

\[
\langle m_e^2 \rangle = \sum_{i=1}^{3} m_i^2 |U_{ei}|^2
\]

only $2 \cdot 10^{-13}$ of decays in this interval

Super-allowed decay
- $Q = 18.6$ keV
- $T_{1/2} = 12.3$ yr

end point

K. Tsumura
The KATRIN experiment started data taking in June 2018, and its goal is to probe the effective neutrino mass with the sensitivity $\sim 0.2$ eV.

The KATRIN detector

- Tritium: $< 2$ eV
- Holmium: $< 225$ eV

The ECHo experiment:

$^{163}_{67}$Ho $\rightarrow ^{163}_{66}$Dy$^* + \nu_e$

$^{163}_{66}$Dy$^* \rightarrow ^{163}_{66}$Dy + $E_C$
The $2\beta$-decays

The $2\beta$ decays: thanks to the mysterious *nuclear pairing force*, some *even-even* nuclei have a rare opportunity to decay to the 2nd nearest neighbors through 2 simultaneous $\beta$ decays (equivalent to the decays of two neutrons).

**necessary conditions:**

- $m(Z, A) > m(Z + 2, A)$
- $m(Z, A) < m(Z + 1, A)$

Maria Goeppert Mayer

First observed by **Michael Moe** in 1987 in Se-82.
The $0\nu2\beta$-decays

★ Theory of the Symmetry of Electrons and Positrons
Ettore Majorana *Nuovo Cim. 14 (1937) 171*

“...there is now no need to assume the existence of antineutron or antineutrinos. The latter particles are indeed introduced in the theory of positive beta-ray emission; the theory, however, can be obviously modified so that the beta-emission, both positive and negative, is always accompanied by the emission of a neutrino.”

The choices of isotopes: a) High Q-value; b) High isotopic abundance; c) Compatibility with detection techniques. Xe-136, Ge-76, Te-130, ...
The effective mass term

The half-life of a $0
\nu
2\beta$-decaying isotope is determined by three factors:

a) phase space factor; b) nuclear matrix element; c) effective mass

$$T^{0\nu}_{1/2} = (G^{0\nu})^{-1} |M^{0\nu}|^{-2} |\langle m \rangle_{ee}|^{-2}$$

- Large uncertainties of NMEs (a factor $\sim 2$)
- Experimental lower bound $> 10^{26}$ year
- A limit on effective mass $< 0.06 \sim 0.20 \text{ eV}$
The reverse is also true

Schechter-Valle THEOREM (1982): if a $0\nu2\beta$ decay occurs, there must be an effective Majorana neutrino mass term.

Four-loop $\nu$ mass:

$$\delta m_\nu \sim 5 \times 10^{-28} \text{ eV}$$

(Duerr, Lindner, Merle, 2011; Liu, Zhou, 2016)

Note: The black box can in principle have many different processes (new physics). Only in the simplest case, which is most interesting, it’s likely to constrain neutrino masses
Vissani’s graph

The effective mass

$$|\langle m \rangle_{ee}| = \sum_i m_i U_{ei}^2$$

Maury Goodman once asked:

An intelligent design?

How likely to fall into the well?
The effective Majorana $\nu$-mass of the neutrinoless double-$\beta$ decays

\[ \langle m \rangle_{ee} = m_1 |U_{e1}|^2 e^{i\rho} + m_2 |U_{e2}|^2 + m_3 |U_{e3}|^2 e^{i\sigma} \]
Contour of the bottom

Let us understand the champagne-bottle profile of the effective $0\nu 2\beta$ mass term in the normal hierarchy case:

$$\langle m \rangle_{ee} = m_1 c_{12}^2 c_{13}^2 e^{i\rho} + m_2 s_{12}^2 c_{13}^2 + m_3 s_{13}^2 e^{i\sigma} = 0$$

$$m_1^2 c_{12}^4 c_{13}^4 + 2m_1 m_2 c_{12}^2 s_{12}^2 c_{13}^4 \cos \rho + m_2^2 s_{12}^4 c_{13}^4 = m_3^2 s_{13}^4$$

Bottom of the Well: not an exact ellipse

The dark well in the normal hierarchy
A bullet structure

\[ |\langle m_{ee}\rangle_{U,L}| = |\overline{m}_{12}\cos^2\theta_{13} \pm m_3\sin^2\theta_{13}| \]

\[ \overline{m}_{12} \equiv \sqrt{m_1^2\cos^4\theta_{12} + \frac{1}{2}m_1m_2\sin^22\theta_{12}\cos\rho + m_2^2\sin^4\theta_{12}} \]

\[ |\langle m_{ee}\rangle_*| = m_3\sin^2\theta_{13} \]

Touching point \( \sim 1 \text{ meV} \)

\( m_1 = m_2\tan^2\theta_{12} \)

\( \rho = \pi \)

Best-fit inputs of current data

Xing, Zhao, arXiv:1612.08538
The threshold or throat

\[ |\langle m \rangle_{ee}|_L = 2|\langle m \rangle_{ee}|_* \]

\[ |\langle m \rangle_{ee}|_L = |\langle m \rangle_{ee}|_* \]

\[ |\langle m \rangle_{ee}|_L = 0 \]

\[ \rho \]

\[ m_1 [\text{eV}] \]

\[ \frac{\partial |\langle m \rangle_{ee}|_L}{\partial \rho} = \frac{m_1 m_2 \sin^2 2\theta_{12} \cos^2 \theta_{13}}{4m_{12}} \sin \rho = 0 \]

\[ \frac{\partial |\langle m \rangle_{ee}|_L}{\partial m_1} = \frac{m_1}{m_3} \sin^2 \theta_{13} \pm \left( \cos^2 \theta_{12} - \frac{m_1}{m_2} \sin^2 \theta_{12} \right) \cos^2 \theta_{13} \neq 0 \]

\[ \frac{\partial |\langle m \rangle_{ee}|_L}{\partial m_1} > 0 \quad \text{for} \quad m_1 < m_2 \tan^2 \theta_{12} \]

\[ \frac{\partial |\langle m \rangle_{ee}|_L}{\partial m_1} < 0 \quad \text{for} \quad m_1 > m_2 \tan^2 \theta_{12} \]
To fall into the well

The three-dimensional parameter space of Vissani Graph

\[ |\langle m \rangle_{ee}| < |\langle m \rangle_{ee^*}| \]

Take it easy!
It is difficult to fall into the well!
What new physics?

**Type (A):** NP directly related to extra species of neutrinos.

**Example 1:** heavy Majorana neutrinos from type-I seesaw

\[-\mathcal{L}_{\text{lepton}} = \overline{l_L}Y_l HE_R + \overline{l_L}Y_\nu \tilde{H} N_R + \frac{1}{2} \overline{N_R} M_R N_R + \text{h.c.}\]

\[\Gamma_{0\nu\beta\beta} \propto \left| \sum_{i=1}^{3} m_i U_{ei}^2 - \sum_{k=1}^{n} \frac{P_{ei}}{M_k} M_A^2 \mathcal{F}(A, M_k) \right|^2\]

In most cases the heavy contribution is negligible

**Example 2:** light sterile neutrinos from LSND anomaly etc

\[\langle m\rangle_{ee}' \equiv \sum_{i=1}^{6} m_i U_{ei}^2 = \langle m\rangle_{ee} (c_{14} c_{15} c_{16})^2 + m_4 (s^*_{14} c_{15} c_{16})^2 + m_5 (s^*_{15} c_{16})^2 + m_6 (s^*_{16})^2\]

In this case the new contribution might be constructive or destructive


**Type (B):** NP has little to do with the neutrino mass issue.

SUSY, Left-right, and some others that I don’t understand
Heavy Majorana neutrinos?

Lepton number violation:
The like-sign dilepton events at the Large Hadron Collider (~ 14 TeV).

Possible collider signature

Collider analogue to $0
\nu\beta\beta$ decay

Dominant channel

$N$ can be produced on resonance
Example (A): R-parity violation

Example (B): R-parity conservation

H.V. Klapdor, hep-ex/9901021
**YES or NO?**

**QUESTION:** are massive neutrinos the Majorana particles?

One might be able to answer **YES** through a measurement of the $0\nu2\beta$ decay or other LNV processes someday, but how to answer with **NO**?

**YES or I don’t know!**

The same question: how to distinguish between Dirac and Majorana neutrinos in a realistic experiment?

**Answer 1:** The $0\nu2\beta$ decay is the only opportunity today.

**Answer 2:** $\nu$’s show different behaviors if nonrelativistic.

**Answer 3:** Atomic physics and cosmology may help a lot.
On sterile neutrinos:

- **sub-eV**
  - active neutrinos

- **sub-eV**
  - sterile neutrinos

- **keV**
  - sterile neutrinos

- **TeV**
  - Majorana neutrinos

- ** ≥ EeV**
  - Majorana neutrinos

**Motivated by**
explaining observed anomalies or solving fundamental issues.

Naturalness / usefulness / testability

- Which type is true or in your belief?
- How many species in each category?
- To what extent is a synergy possible?

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**The 3rd law of progress in theoretical physics:**

You may use *any* degrees of freedom you like to describe a physical system, but if you use the wrong ones, you will be sorry —— S. Weinberg 83.
\[
\begin{pmatrix}
\nu^e \\
\nu^\mu \\
\nu^\tau \\
\nu^x \\
\nu^y \\
\nu^z
\end{pmatrix}
= \mathcal{U}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\nu_5 \\
\nu_6
\end{pmatrix}
\]

\[
\mathcal{U} = \begin{pmatrix}
1 & 0 \\
0 & U_0
\end{pmatrix}
\begin{pmatrix}
A & R \\
S & B
\end{pmatrix}
\begin{pmatrix}
V_0 & 0 \\
0 & 1
\end{pmatrix}
\]

Full parametrization:
- 15 rotation angles
- 15 phase phases

arXiv:1110.0083
Concluding remark

All of us expect the massive neutrinos to be the Majorana particles. If this expectation comes true someday thanks to the $0\nu2\beta$ decays, then we have to have some right/good ideas to probe Majorana phases.

Extremely difficult? Even impossible? Why we meet here in Okinawa? History tells us: the fool didn’t know it’s impossible, so he did it and...

C.S. Wu: It is easy to do the right thing once you have the right ideas.

L.C. Pauling: The best way to have a good idea is to have a lot of ideas.